Phase Unwrapping by Minimizing Kikuchi Free Energy

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Abstract

Phase unwrapping in 2-dimensional topologies is an important problem that has several applications in radar and satellite imaging. Sum product algorithm (belief propagation) gives excellent results for the phase unwrapping problem. In this work, we present a gradient smoothing technique that uses higher order surface models to produce very smooth surfaces and report an improvement in the solution obtained.
In a recent important work, Yedidia et.al have showed the theoretical connections between belief propagation algorithms and free energy in statistical physics. Based on this, we present a model that uses the Kikuchi technique to compute better posterior marginals than those produced by sum product algorithm.

1 Introduction

Given real-valued observations on a 2-dimensional grid that are measured modulus a known wavelength, the goal of phase unwrapping is to infer the original, unwrapped surface. It is usually assumed that the surface to be estimated varies smoothly in some topology (without such an assumption the problem is ill-posed). Some approaches to solving this problem include integer programming/network flow based methods, branch cut techniques and algorithms using least squares estimates[3].

An approach to phase unwrapping would be to infer the relative wrappings or the integer number of relative shifts($\mathcal{S}$) between neighboring measurements, subject to an *a priori* preference for smooth surfaces. If neighbouring measurements are ϕ_i and ϕ_{i+1}, the true gradient between them is $\phi_{i+1} - \phi_i - \mathcal{S}$. The inferred vector field is a gradient field of a surface and, hence, must obey the zero curl constraint which asserts that the curl along any closed path be zero. This constraint guarantees that the resulting surface will be independent of the direction of integeration. Without loss of generality, we restrict the integer shifts to be -1, 0 or 1 (*i.e.*, wavelength =1). A preference for smooth surface can be obtained by asuming a Gaussian prior on adjacent heights.

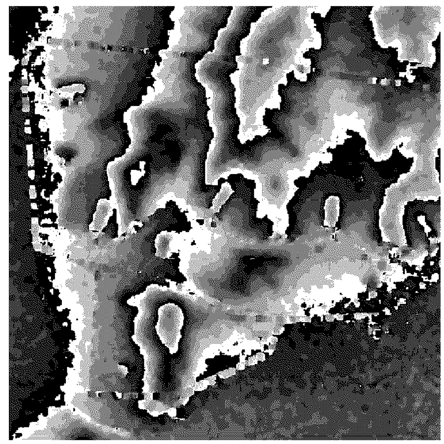

Figure 1: A wrapped topograhic map (courtesy: Sandia laboratories, New Mexico)

Exact inference in a 2D grid is intractable due to the very large number of paths that need to be evaluated and readily calls for approximate inference techniques.

2 Sum product algorithm

Let $\phi_{i,j} \in [0,1]$ be the phase value at i,j. Let $a_{i,j} \in \mathcal{I}$ be the unknown shift between points i,j and $i,j+1$. So, the difference in the unwrapped values at pixels $i,j+1$ and i,j is $\phi_{i,j+1} - \phi_{i,j} - a_{i,j}$. Similarly, let $b_{i,j} \in \mathcal{I}$ be the unknown shift between points at i,j and $i+1,j$.

The quantity $a(x,y)+b(x+1,y)-a(x,y+1)-b(x,y)$ is a measure of curl at location (x,y) and is constrained to 0. Note that if the sum of shifts around every $2x2$ loop is zero, then the zero curl constraint is satisfied. Since any valid configuration of the shifts must obey this constraint we define the prior probability on the shifts by $\delta(curl)$, which evaluates to 1 if its argument is 0 and 0 otherwise.

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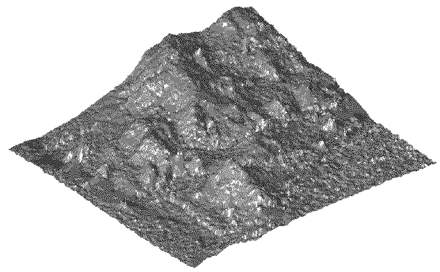
(a)



(b)

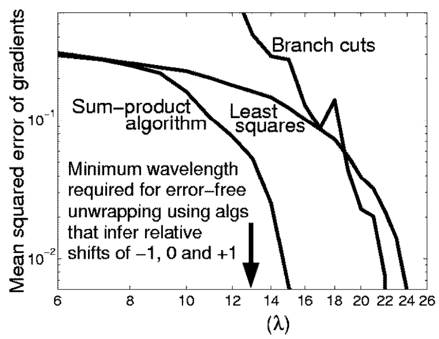


Figure 2: (a) After 10 iterations of sum-product algorithm on the wrapped map shown in Fig.1. Hard decisions were made for shift variables and the resulting gradients were integrated. (b) Comparison of performance of sum-product algorithm with other methods

Assuming a Gaussian likelihood for the gradients,

$$p(\phi|a,b) \propto \prod_{i,j} \Big(\exp[-(\phi_{i,j+1} - \phi_{i,j} - a_{i,j})^2/2\sigma^2] \cdot \exp[-(\phi_{i+1,j} - \phi_{i,j} - b_{i,j})^2/2\sigma^2] \Big)$$

This likelihood favours shifts that result in small change in gradient between adjacent pixels. σ can be interpreted as the surface variability factor. Phase unwrapping consists of making inferences about the a's and b's in the above model. The prior and the likelihood function can be used to draw a factor graph[5] on which the sum-product algorithm can be applied (see[2, 4] for more details). Inference in this model involves finding the posterior marginals (revised beliefs $p(a,b|\phi)$) corresponding to every shift variable.

Fig.2b shows the logarithm of the mean squared error in the estimated gradients of the reconstructed surface as a function of the wrapping wavelength(λ) for the data in Fig.1. Here, the surface obtained using least squares is assumed to be the ground truth. As $\lambda \to 0$, unwrapping becomes impossible and as $\lambda \to \infty$, unwrapping becomes trivial (since no wrappings occur) and so algorithms have waterfall-shaped curves. Clearly, sum-product algorithm obtaines significantly lower error rates compared to other techniques.

3 Higher order surface models

In [1, 2, 4] the constraint on smoothness was enforced between every pair of adjacent pixels. This corresponds to first order smoothing and is limited to a small neighbourhood.

A very natural extension to this would be to limit the variations in near by gradients along the x and y direction. The notion of gradient smoothing has direct link to the vector Taylor series expansion of the height $h(x+\Delta x, y+\Delta y)$. Here an x direction shift is smoothed by the neighbouring x directional shift and similarly for the y direction shift. The likelihood based on the local measurements as computed in [2] is weighted by the smoothing factor. This factor for a x-direction shift variable at (i,j) is given by

$$\sum_{a_{i-1,j}a_{i+1,j}} \Big(\exp[-(\Delta x_{i+1,j} - a_{i+1,j} - \Delta x_{i,j} + a_{i,j})^2/2\gamma^2] \cdot \exp[-(\Delta x_{i,j} - a_{i,j} - \Delta x_{i-1,j} + a_{i-1,j})^2/2\gamma^2] \Big)$$

where $\Delta x_{i+1,j} = \phi_{i+1,j+1} - \phi_{i+1,j}$ and γ corresponds to the variance in the gradient. When the smoothing variance γ is near zero the model favours a plane and when it is very high the model doesn't discriminate gradient changes resulting in the estimates produced by sum-product algorithm.

This additional smoothness factor can be incorporated in to the message passing frame work described in [2] by adding extra messages for the gradients. After an initial sweep using the sum product algorithm, we smooth the gradients along the x direction gradients with the neighbouring x direction gradients (the ones above and below) and similarly for the gradients along the y direction.

In Fig.3 we compare the performance of this new gradient smoothing technique with sum-product algorithm using a first order likelihood. When the smoothness variance γ is low, reconstruction error is high (as the model doesn't allow for any variability in the surface) and when γ gets very large the effects of gradients are ignored and the results converge with the estimates of the sum-product algorithm.

It is interesting to note that the reconstruction error decreases for a while before converging to the sum-product estimates. Selecting the right smoothness variance γ is crucial and based on our experiments a

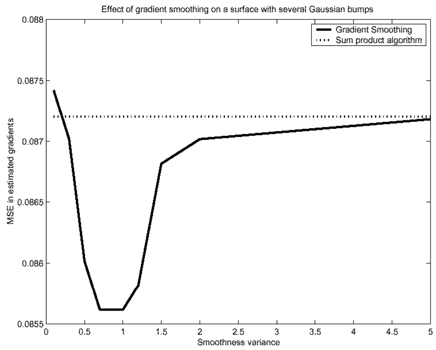


Figure 3: Effect of variance on MSE

good rule of thumb is to compute γ from the wrapped data. This estimate of variance does not necessarily correspond to the minimum error, but based on our experiments it usually corresponds to a solution that is not inferior to the sum-product algorithm.

4 Kikuchi approximation technique

Sum-product algorithm gives excellent results for the phase unwrapping problem and out performs many existing techniques [2]. Recently, it has been proved that the fixed points of belief propagation(BP) correspond to the stationary points of the Bethe free energy[7]. This important result connects BP to approximate variational inference [1] and provides the option of minimizing the Bethe free energy directly. Further, this opens the possibility to minimize more accurate approximations to free energy such as Kikuchi free energy to obtain better solutions.

Kikuchi free energy (cluster variational method), is computed by approximating free energy over clusters of nodes. It is the sum of the free energies of the basic clusters less the over-counted cluster intersections. It subsumes mean field and Bethe approximations as special cases with cluster size 1 and 2. Interestingly, the stationary points of Kikuchi approximation correspond to the fixed points of *generalized belief propagation*, a generalization of the standard BP[6].

Our work in [1] corresponds to Kikuchi approximation with cluster size 1; This work and [2, 4], correspond to cluster size 2. Generalized belief propagation operates by passing messages between clusters of shift variables. Increasing the basic cluster size yields better approximation to the posterior marginals. However, arbitrarily large clusters can pose problem as the size of state space increases exponentially with the cluster size. But, the state space for phase unwrapping problem can be significantly sized down by only considering cluster configurations that satisfy the zero curl constraint. Alternatively, we can minimize the Kikuchi approximate free energy of the system directly.

5 Conclusion

We proposed a new gradient smoothing technique that improves the estimates produced by sum product algorithm on synthetic surfaces with several Gaussian bumps. As noted, the theoretical connection between belief propagation algorithms and free energy formulation provides scope for more powerful algorithms. We are currently experimenting Kikuchi approximation technique and generalized BP with cluster size 4.

Acknowledgments

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