UNWRAPPING PHASES BY RELAXED MEAN FIELD INFERENCE

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ABSTRACT

Some types of medical and topographic imaging device produce images in which the pixel values are "phase-wrapped", *i.e.*, measured modulus a known scalar. *Phase unwrapping* can be viewed as the problem of inferring the number of shifts between each and every pair of neighboring pixels, subject to an *a priori* preference for smooth surfaces, and subject to a zero curl constraint, which requires that the shifts must sum to 0 around every loop. We formulate phase unwrapping as a mean field inference problem in a probability model, where the prior favors the zero curl constraint. We compare our mean field technique with the least squares method on a synthetic 100×100 image, and give results on a larger 512×512 image.

1. INTRODUCTION

The problem of inferring unwrapped values from phase-wrapped measurements is a fundamental problem in signal processing, which seems as though it should be solvable, and yet which remains unsolved. Phase unwrapping has applications in a variety of sensory modalities, including magnetic resonance imaging [1] (see Fig. 1a) and interferometric synthetic aperture radar (SAR) [2] (see Fig. 1b).

In many applications, multiple phase-wrapped measurements are available and there is *a priori* knowledge about the probable relationships between the phase-*unwrapped* values – *e.g.*, they vary smoothly in some topology. (Without prior knowledge, the wrapped image itself provides an error-free guess at the unwrapped image.) Exact inference in a 2-dimensional topology is generally intractable because the number of distinct paths connecting two points is at least equal to the width of the image, and all combinations of elevations along these paths should be examined. In fact, if phase unwrapping is cast as a "minimum L^0 norm problem" in integer programming, it turns out to be NP-hard [3].

Approaches to solving the phase unwrapping problem include least squares estimates (these are *not* MMSE estimates) [4, 2, 5, 1], integer programming methods [6, 3] and branch cut techniques [7].

Inferring the unwrapped values is equivalent to inferring the relative number of shifts between each and every pair of neighboring pixels. These integers can be combined with the observations to produce a gradient field, which can then be integrated to reconstruct the unwrapped image. However, only a subset of the possible configurations of these shifts

(a)

(b)

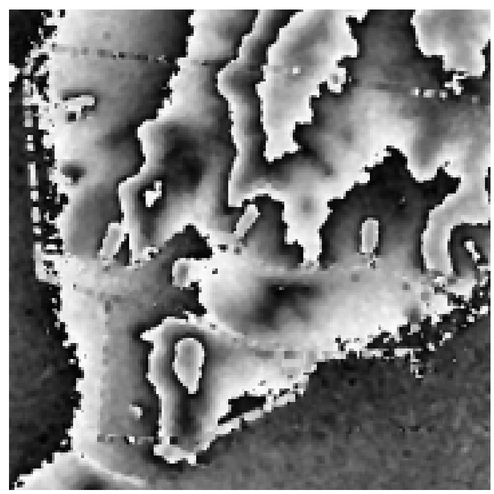

Fig. 1. Phase-wrapped images from (a) magnetic resonance imaging data (courtesy of Z.-P. Liang) and (b) synthetic aperture radar data (courtesy of Sandia National Laboratories, New Mexico). Pixel values close to 0 are painted white, whereas pixel values close to 1 (the wrapping wavelength) are painted black.

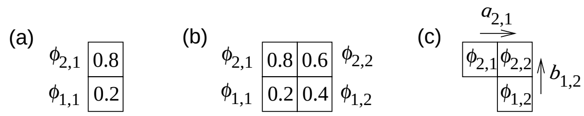

Fig. 2. Phase measurements in small image patches. From (a) it appears that a shift occurred between points $1,1$ and $2,1$. From (b), it appears that a shift probably did *not* occur between points $1,1$ and $2,1$. (c) The phase at 2,2 can be predicted from the phases at 2,1 and 1,2, plus the shifts $a_{2,1}$ and $b_{1,2}$.

will lead to valid gradient fields. In particular, the sum of the shifts around any loop in the image must be zero. We refer to this constraint as the *zero curl constraint*.

We formulate phase unwrapping as a mean field inference problem in an relaxed probability model, where the prior favors shifts that satisfy the zero curl constraint. We relax the prior by introducing a temperature parameters. The preference for shifts that satisfy the zero curl constraint is weakened at high temperatures. As the temperature is decreased to zero (annealing), the model settles to a consistent configuration of the shifts.

2. RELAXED GRADIENT FIELD MODEL

Let $\phi_{i,j} \in [0,1)$ be the phase value at i,j. (We assume that measurements are taken *modulus* 1 – *i.e.*, the wavelength is 1.) Let $a_{i,j} \in \mathcal{I}$ be the unknown shift between points i,j and $i,j+1$. So, the difference in the unwrapped values at pixels $i,j+1$ and i,j is $\phi_{i,j+1} - \phi_{i,j} - a_{i,j}$. Similarly, let $b_{i,j} \in \mathcal{I}$ be the unknown shift between points at i,j and $i+1,j$.

Consider the two patches of image shown in Fig. 2. From Fig. 2a, the difference in the unwrapped values at $1,1$ and $2,1$ is $0.8 - 0.2 - a_{1,1}$. Assuming the values are more likely to be closer together than further apart, we decide that $a_{1,1} = 1$, so that $0.8 - 0.2 - a_{1,1}$ is as close to 0 as possible.

We can make these local decisions for every neighboring pair of points in a large image, but the resulting set of shifts will not satisfy the constraint of summing to zero around every loop. If we make local decisions for the patch in Fig. 2b, then we decide that $a_{1,1} = 1$, $b_{1,2} = 0$, $a_{2,1} = 0$ and $b_{1,1} = 0$. The sum of these shifts around a counter-clockwise loop is $a_{1,1} + b_{1,2} - a_{2,1} - b_{1,1} = 1$, giving a *curl violation*. We can fix this curl violation by changing one or more of the shifts, at the cost of not keeping the unwrapped pixel differences as close to zero as possible.

Notice that if the the sum of the shifts around every 2×2 loop is zero, then the sum of the shifts around any loop is zero. So, the 2×2 loops provide a sufficient set of constraints.

To choose the form of the above cost, we develop a probability model of the shifts and the observed phases. We choose a prior for the shifts that favors shifts that satisfy the zero curl constraint:

$$p(a,b) \propto \prod_{i,j} \exp[-(a_{i,j} + b_{i,j+1} - a_{i,j+1} - b_{i,j})^2/T]$$

To incorporate a preference for smooth surfaces, we restrict the values of the a's and b's to be in $\{-1,0,1\}$.

The density of the observed phase measurements can be formulated recursively. As shown in Fig. 2c, the phase at 2,2 can be predicted from the phases at 2,1 and 1,2, plus the shifts $a_{2,1}$ and $b_{1,2}$. The prediction from 2,1 is $\phi_{2,1} + a_{2,1}$, while the prediction from 1,2 is $\phi_{1,2} + b_{1,2}$. The average prediction is $(\phi_{2,1} + a_{2,1} + \phi_{1,2} + b_{1,2})/2$. Assuming a Gaussian likelihood we obtain the general form

$$p(\phi|a,b) \propto \prod_{i,j} \Big(\exp[-(\phi_{i,j+1} - \phi_{i,j} - a_{i,j})^2/2\sigma^2] \cdot \exp[-(\phi_{i+1,j} - \phi_{i,j} - b_{i,j})^2/2\sigma^2] \Big)$$

The joint distribution $p(a,b,\phi) = p(a,b)p(\phi|a,b)$ is

$$p(a,b,\phi) \propto \prod_{i,j} \exp[-(a_{i,j} + b_{i,j+1} - a_{i,j+1} - b_{i,j})^2/T] \cdot \prod_{i,j} \Big(\exp[-(\phi_{i,j+1} - \phi_{i,j} - a_{i,j})^2/2\sigma^2] \cdot \exp[-(\phi_{i+1,j} - \phi_{i,j} - b_{i,j})^2/2\sigma^2] \Big)$$

2.1. Temperature

The temperature T allows the prior to be relaxed. For $T \to \infty$, the zero curl constraint is completely relaxed. For $T \to 0$, only shifts that satisfy the zero curl constraint have non-vanishing probability.

3. MEAN FIELD INFERENCE

Exact inference (*e.g.*, computing $p(a_{i,j}|\phi)$) in the above model is intractable. So, we use a mean field approximation (c.f. [8]).

We approximate $p(a,b|\phi)$ with a factorized distribution,

$$q(a,b) = \prod_{i,j} q(a_{i,j}) q(b_{i,j}). \tag{1}$$

We parameterize the q-distribution as follows:

$$q(a_{i,j} = k) = \alpha_{i,j,k}, \quad q(b_{i,j} = k) = \beta_{i,j,k}, \tag{2}$$

where we require $\sum_{k=-1}^{1} \alpha_{i,j,k} = 1$ and so on. The α's and β's are *variational parameters* (c.f. [8]).

To bring q "close" to p, we would like to minimize the relative entropy,

$$D = \sum_{a,b} q(a,b) \log \frac{q(a,b)}{p(a,b|\phi)}. \tag{3}$$

However, this quantity contains $p(a, b|\phi)$ for which we do not have a simple, closed form expression.

Subtracting $\log p(\phi)$ (which does not depend on the variational parameters) from the above relative entropy, we obtain a cost function that *can* be easily minimized:

$$
\begin{aligned}
F &= D - \log p(\phi) \\
&= \sum_{a,b} q(a,b) \log \frac{q(a,b)}{p(a,b,\phi)} \\
&= \ldots \\
&= \sum_{i,j} \Big(\sum_{k=-1}^{1} \alpha_{i,j,k} \log \alpha_{i,j,k} + \sum_{k=-1}^{1} \beta_{i,j,k} \log \beta_{i,j,k} \Big) \\
&\quad + \frac{1}{T} \sum_{i,j} \sum_{k,l,m,n} \alpha_{i,j,k} \beta_{i,j+1,l} \alpha_{i+1,j,m} \beta_{i,j,n} (k+l-m-n)^2 \\
&\quad + \frac{1}{2\sigma^2} \sum_{i,j} \Big(\sum_{k=-1}^{1} \alpha_{i,j,k} (\phi_{i,j+1} - \phi_{i,j} - k)^2 \\
&\quad + \sum_{k=-1}^{1} \beta_{i,j,k} (\phi_{i+1,j} - \phi_{i,j} - k)^2 \Big). \qquad (4)
\end{aligned}
$$

For the results presented below, we use a conjugate gradient optimizer (including Langrangian constraints to ensure that $\sum_{k=-1}^{1} \alpha_{i,j,k} = 1$ and so on).

4. RESULTS

We present results on two images. In the first case, we synthesized the original image (surface), so we know the "ground truth" and can easily compare our method with the standard least squares technique [4, 2, 5, 1]. In the second case, we present results on unwrapping the Sandia image.

4.1. Synthetic data

Fig. 3a shows the phase-wrapped image produced from our synthetic data. After minimizing F using 20 iterations of conjugate gradients, while annealing the temperature from a high value to a low value, we obtained a set of shift probabilities (α's and β's from above). For each pair of pixels, we picked the shift that had highest probability. The resulting set of shifts satisfied the zero curl constraint. From the shifts, we obtained a gradient field and integrated it to obtain the surface shown in Fig. 3b. This surface matches the original surface perfectly.

We applied the least squares method to the wrapped data in Fig. 3a and obtained the surface shown in Fig. 3b. In contrast to our mean field method, the least squares method produces ridge-like artifacts.

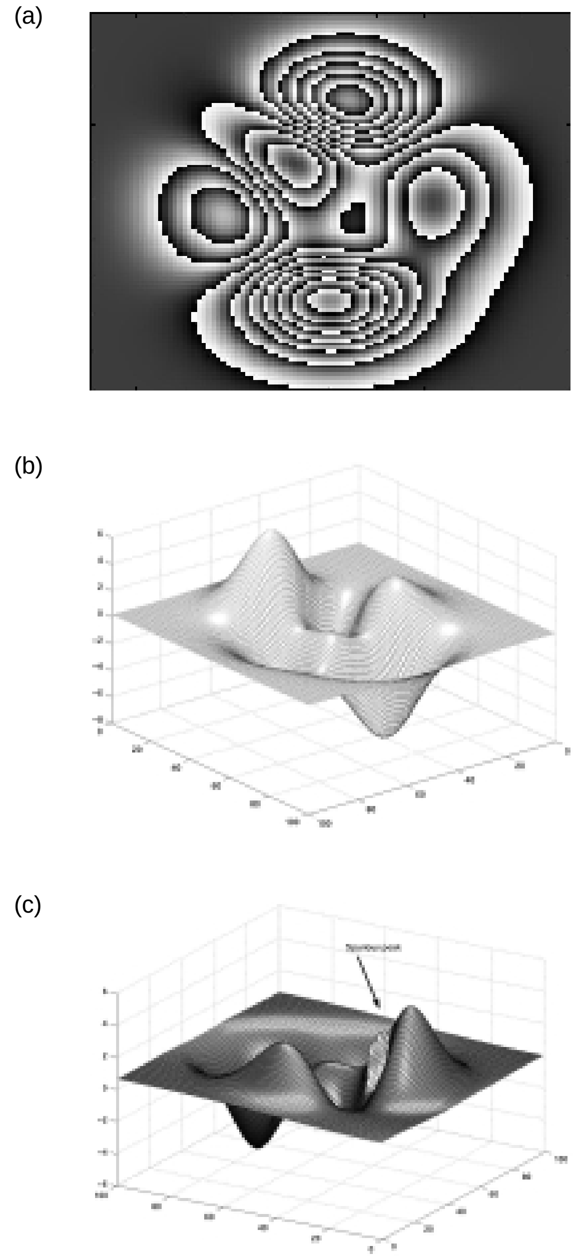

Fig. 3. (a) A 100×100 wrapped image. (b) Unwrapped surface produced by our mean field technique. (c) Unwrapped surface produced by the least squares method.

4.2. Sandia data

After minimizing F using as input the 512×512 phase-wrapped image from the Sandia National Laboratories, New Mexico (Fig. 1b), we found that there were still some zero curl violations. Using the shifts to produce a "gradient field" produces a "gradient field" that violates the zero curl constraint. So, we used our method as a preprocessor for the least squares method, obtaining the surface shown in Fig. 4

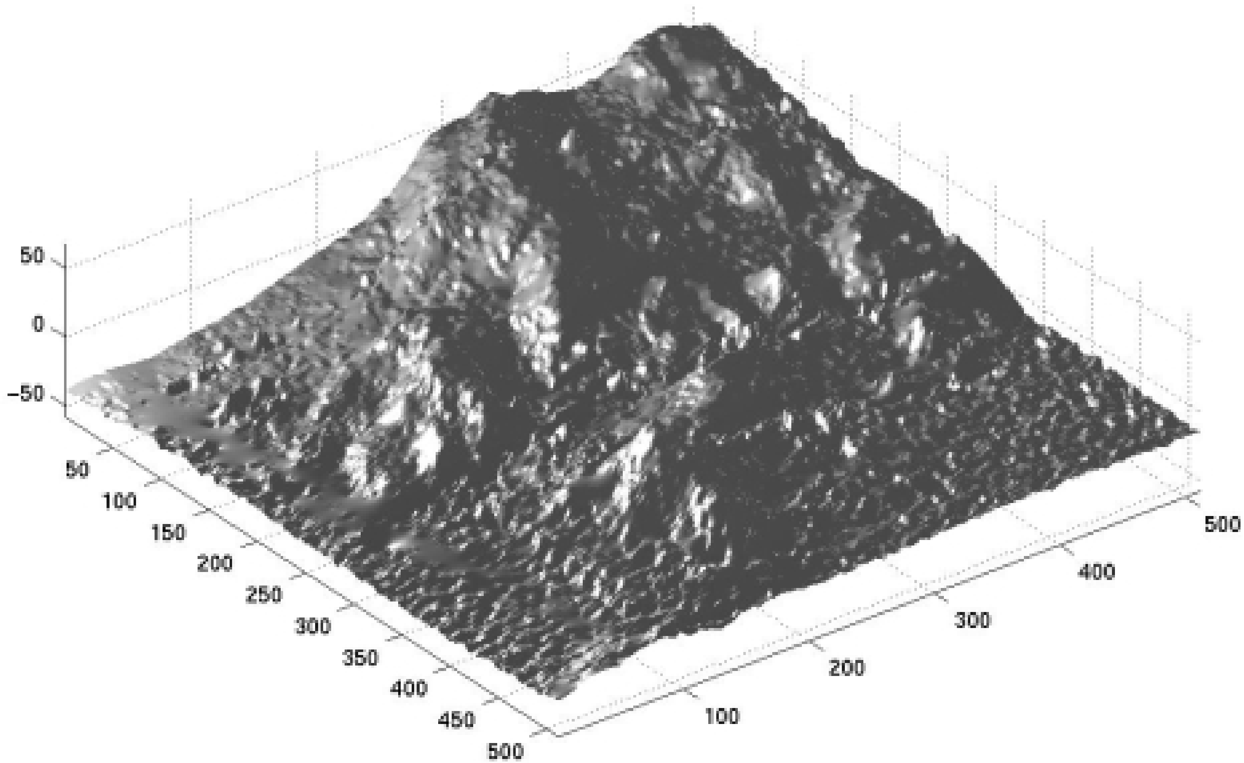

Fig. 4. Unwrapped surface produced by our mean field method applied to the 512×512 Sandia data shown in Fig. 1b.

5. CONCLUSIONS

We introduced a relaxed mean field technique for phase unwrapping and we illustrated that it can perform better than the least squares technique on a 100×100 image.

From our results on the Sandia data, it appears the mean field method leave some some of the zero curl constraints violated. To overcome the ensuing problem of trying to integrate an invalid gradient field, we used the mean field technique as a preprocessor for the least squares method.

6. ACKNOWLEDGEMENTS

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7. REFERENCES

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